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A Hopf-Algebra Approach to Inner Plethysm

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We use the Hopf algebra structure of the algebra of symmetric functions to study the Adams operators of the complex representation rings of symmetric groups, and we give new proofs of all of Littlewood's formulas for inner plethysm. We also study the Adams operations for orthogonal and symplectic group characters.

1. INTRODUCTION

It is well known that the direct sum $\bigoplus_{n \geqslant 0} R(S_n)$ of the representation rings of all symmetric groups is endowed with the structure of a self-dual Hopf algebra [40, 9, 8], the product being given by the Frobenius outer tensor product

$$R(S_p) \times R(S_q) \ni ([\rho], [\eta]) \mapsto [\rho] \cdot [\eta] = [\mathrm{ind}_{S_p \times S_q}^{S_{p+q}} \rho \times \eta] \in R(S_{p+q})$$

and the coproduct being defined as

$$R(S_n) \ni [\rho] \mapsto \Delta([\rho]) = \sum_{p+q=n} [\mathrm{res}_{S_p \times S_q}^{S_n} \rho].$$

In this paper, we systematically exploit this situation to compute with the individual λ-ring structures of the $R(S_n)$ (for which the product is induced by the ordinary tensor product of S_n-representations). In particular, we give a full description of the adjoints of the Adams operators, in terms of coproducts and ordinary plethysms of symmetric functions (Section 3).

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This leads to direct proofs of Littlewood's formulas for the stable reduction of symmetrized inner Kronecker squares, and to analogous formulas for higher powers (Section 4). These computations are closely related to root-number functions in symmetric groups, and to various symmetric generalizations of the cyclotomic identity. The same techniques allow us to obtain the branching rules for the embeddings $GL(n) \supset S_n$ and $GL(n-1) \supset S_n$ in the form of simple reciprocity laws (Section 5). As a further illustration of the formalism, we also give proofs of Littlewood's formulas for products and plethysms with characters of orthogonal and sympletic groups and give a general expression for the Adams operators of the representation rings of these groups (Section 6).

Parts of these results were announced in [33], where formulas (4) and (5) (cf. Theorems 4.4 and 4.6 of the present paper) were incorrectly stated.

The problem of finding simple proofs of Littlewood's formulas was suggested in [28, p. 64].

2. Background

Our notations for symmetric functions (of an infinite, countable set of indeterminates $X = \{x_1, x_2, ...\}$) will be the same as in [27], except for the following minor changes. We denote by $\mathrm{Sym}(X) := \bigoplus_n \mathrm{Sym}^n(X)$ the $\mathbf{Z}$-algebra of symmetric functions over X, graded by total degree. The standard scalar product on $\mathrm{Sym}(X)$ (for which the Schur functions form an orthonormal basis) is denoted by $(\cdot, \cdot)$ rather than by $\langle \cdot, \cdot \rangle$, the symbol $\langle \cdot \rangle$ being reserved for an other use. Also, for a symmetric function F, we denote by D_F instead of $D(F)$ the adjoint of the linear operator $G \mapsto FG$. The generating series for complete and elementary functions are

$$\sigma_z(X) = \sum_{n \geqslant 0} z^n h_n(X) = \prod_{i \geqslant 1} (1 - zx_i)^{-1}$$

and

$$\lambda_z(X) = \sum_{n \geqslant 0} z^n e_n(X) = \prod_{i \geqslant 0} (1 + zx_i).$$

A partition can be described by the sequence of its parts, arranged in decreasing order or by the multiplicities of the parts. If $\alpha = (\alpha_1, \alpha_2, ...) = (1^{a_1} 2^{a_2} \cdots)$, then we call the sequence $a := (a_1, a_2, ...)$ the *cycle type* of α.

As already mentioned in the Introduction there are two multiplicative structures on $\mathrm{Sym}(X)$. The ordinary or *outer product*, i.e., the usual multiplication of symmetric functions, and the *internal* or *inner product* which is denoted by $*$. We recall that the latter can be defined on the basis p_λ of

$\mathbf{Q} \otimes \mathrm{Sym}(X)$ by $p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda$, and that if χ and η are characters of a symmetric group S_n, then

$$\mathrm{ch}(\chi\eta) = \mathrm{ch}(\chi) * \mathrm{ch}(\eta),$$

where ch is the Frobenius characteristic map [27, I.7; 18], ch: $\oplus_n R(S_n) \to \mathrm{Sym}(X)$ (we will often identify representations with their characters).

The algebra $\mathrm{Sym}(X)$ is naturally endowed with two coproducts Δ and δ, which are more easily defined using the standard λ-ring structure and the identification

$$\theta: F(X)\, G(Y) \mapsto F \otimes G \in \mathrm{Sym}(X) \otimes \mathrm{Sym}(X),$$

where Y is a second countable set of indeterminates, independent from X. Then we define

$$\Delta F = \theta[F(X+Y)]$$

and

$$\delta F = \theta[F(XY)].$$

This amounts to specifying that

$$\Delta p_i = p_i \otimes 1 + 1 \otimes p_i$$

$$\delta p_i = p_i \otimes p_i$$

and that Δ and δ are algebra morphisms for the ordinary product. In fact, it can be shown that both are also morphisms for the internal product, and a fundamental fact is that they are respectively adjoints to the ordinary and internal product:

2.1. THEOREM. *For any $F, G \in \mathrm{Sym}(X)$,*

$$(\Delta F, G \otimes H) = (F, GH)$$

$$(\delta F, G \otimes H) = (F, G * H).$$

The same is of course true for the *iterated coproducts* Δ^r and δ^r, again defined via the identification

$$\theta_r : F_1(X_1) \cdots F_r(X_r) \mapsto F_1 \otimes \cdots \otimes F_r,$$

by

$$\Delta^r F = \theta_r F(X_1 + \cdots + X_r), \qquad \text{and} \qquad \delta^r F = \theta_r F(X_1 \cdots X_r).$$

In this context it is often useful to consider (iterated) multiplication as an operator $\mathrm{Sym}(X)^{\otimes r} \to \mathrm{Sym}(X)$. Thus we will denote the r-fold outer (inner) products by M_r (resp. m_r) and we will suppress the suffix in the case $r=2$.

The plethysm $F \circ G$ [27, I.8] of G by F is here denoted by $F(G)$, F being interpreted as an operator of the λ-ring $\mathrm{Sym}(X)$. For each $G \in \mathrm{Sym}(X)$ there is an algebra homomorphism

$$R(G) : \mathrm{Sym}(X) \to \mathrm{Sym}(X), \qquad F \mapsto F(G).$$

We will write $Q(G)$ for its adjoint. For further use we note two properties of these mappings

$$R(G+H) = M \circ R(G \otimes H) \circ \Delta, \qquad R(G \cdot H) = M \circ R(G \otimes H) \circ \delta,$$

where $R(G \otimes H) := R(G) \otimes R(H)$. The Adams operations $F \mapsto p_k(F)$ of this λ-ring will be also denoted by ψ_k, and the adjoints of the linear operators ψ_k will be denoted by ϕ_k. It is easily shown that ψ_k is a homomorphism for the outer product and that ϕ_k is one for both outer and inner products. A detailed study of the ϕ_k can be found in [13].

Following Littlewood [25], we define now a second kind of plethysm on the ring of symmetric functions, called inner plethysm, and corresponding, via the Frobenius characteristic map, to symmetrized inner tensor products of symmetric group representations [10]. The operators of inner plethysm will be written with hats, to be distinguished from those of ordinary (or *outer*) plethysm. If $P = \mathrm{ch}(\chi_\rho)$ where χ_ρ is the character of the representation ρ, we define $\hat{e}_k(P)$ as $\mathrm{ch}(\bigwedge^k(\rho))$ where $\bigwedge^k(\rho)$ is the kth exterior power of the representation ρ. Now, if $F = \sum_\lambda c_\lambda e_\lambda$, then

$$\hat{F}(P) = \sum_\lambda c_\lambda \hat{e}_\lambda(P),$$

where $\hat{e}_{(\lambda_1, \ldots, \lambda_r)}(P) = \hat{e}_{\lambda_1}(P) * \hat{e}_{\lambda_2}(P) * \cdots * \hat{e}_{\lambda_r}(P)$. As above we set for the Adams operations $\hat{\psi}_k(F) := \hat{p}_k(F)$, and we denote by $\hat{\phi}_k$ the adjoint of $\hat{\psi}_k$ (cf. [9, 39, 32]). It is easily shown that $\hat{\phi}_k$ is also a morphism for the ordinary product.

We may, via the characteristic map, also consider $\hat{\psi}_k$ and $\hat{\phi}_k$ as operators on the ring of complex-valued class-functions of the symmetric group. Then we have for any class-function χ

$$[\hat{\psi}_k(\chi)](\pi) = \chi(\pi^k), \qquad [\hat{\phi}_k(\chi)](\pi) = \sum_{\rho : \rho^k = \pi} \chi(\rho).$$

If we take the trivial character ι, then $\hat{\phi}_k(\iota)$ counts the number of kth roots in the group. Hence we may call $\hat{\phi}_k(\iota)$ the kth *root number function*. Both equations can be used to define operators on the class functions of an arbitrary finite group. Frobenius showed that these operators are (by restriction) operators on the ring of virtual characters [6].

Inner plethysm and the internal product make the additive group $\mathrm{Sym}^n(X)$ of symmetric functions of weight n into a λ-ring, isomorphic to the representation ring $R(S_n)$ of the symmetric group S_n. The individual structures of the rings $\mathrm{Sym}^n(X)$ can be glued together, to define a λ-ring structure on the ring $\mathrm{Sym}(X)^\wedge$ of symmetric formal power series, the unit element of $*$ being the series $\sigma_1 = \sum_n h_n$ (recall that $F * G = 0$ if F and G are homogeneous of different weights, and that h_n is the image by ch of the trivial character of S_n).

We also make use of the linear operator defined on the basis of Schur functions by

$$\Gamma(z) : \mathrm{Sym}(X) \to \mathbf{Z}(X) \otimes \mathrm{Sym}(X)^\wedge, \qquad s_\lambda \mapsto \sum_{p \in \mathbf{Z}} z^p s_{(p,\lambda)}$$

where for $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbf{Z}^r$, $(p, \lambda) = (p, \lambda_1, ..., \lambda_r) \in \mathbf{Z}^{r+1}$. This operator is a particular case of the so-called *vertex operators* (see, e.g., [11], and [3, 37] for applications to symmetric group characters), and its fundamental property is the factorization

$$\Gamma(z) F = \sigma_z \, D_{\lambda_{-1/z}} F = \sigma_z(X) \, F(X - \tfrac{1}{z}).$$

In the following, the image of a symmetric function F by the operator $\Gamma(1)$ will be denoted by $\langle F \rangle$. Let us note that $D_{\lambda_{-1}}$ is a ring automorphism of $\mathrm{Sym}(X)$ whose inverse is D_{σ_1}. Moreover, in terms of λ-rings,

$$(D_{\lambda_{-1}} F)(X) = F(X-1), \qquad (D_{\sigma_1} F)(X) = F(X+1).$$

From this, for example, it follows that

$$D_{\lambda_{-1}} \circ R(G) = R(D_{\lambda_{-1}} G), \qquad D_{\sigma_1} \circ R(G) = R(D_{\sigma_1} G).$$

The operator $\Gamma(1)$ accounts for example for the fact that the value $\chi^\lambda(\rho)$ of an irreducible character of S_n on a conjugacy class $\rho = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$ is given by a polynomial $\Xi_{\lambda_2, ..., \lambda_r}(\alpha_1, \alpha_2, ...)$ independent of the first part λ_1 of λ (see [12]). Indeed, putting $\mu = (\lambda_2, ..., \lambda_r)$, we have

$$\begin{aligned} \chi^\lambda(\rho) &= (\langle s_\mu \rangle, p_\rho) = (\sigma_1 \, D_{\lambda_{-1}} s_\mu, p_\rho) = (D_{\lambda_{-1}} s_\mu, D_{\sigma_1} p_\rho) \\ &= \sum_{k \geq 0} (-1)^k \left(s_{\mu/1^k}, \prod_{i \geq 1} (1 + p_i)^{\alpha_i} \right) \end{aligned}$$

which is equivalent to Gamba's formula [7].

Also, it can be shown that an internal product of the form $\langle F\rangle * \langle G\rangle$, where $F, G \in \mathrm{Sym}(X)$, can be expressed as a finite linear combination of series of the form $\langle H\rangle$, $H \in \mathrm{Sym}(X)$. More precisely:

2.2. THEOREM [37]. *Let (u_λ) and (v_λ) be adjoint bases of* $\mathrm{Sym}(X)$ *or of* $\mathbf{Q} \otimes \mathrm{Sym}(X)$. *Then, for any $F, G \in \mathrm{Sym}(X)$,*

$$\langle F\rangle * \langle G\rangle = \left\langle \sum_{\lambda,\mu} D_{u_\lambda} F\, D_{u_\mu} G\, D_{\sigma_1}(v_\lambda * v_\mu) \right\rangle.$$

Taking $s_\lambda := u_\lambda = v_\lambda$ this yields

2.3. COROLLARY [26] (see [37, 1.1] for a proof).

$$\langle F\rangle * \langle G\rangle = \left\langle \sum_{\lambda\mu\nu} (D_{s_\nu} D_{s_\lambda} F)(D_{s_\nu} D_{s_\mu} F)(s_\lambda * s_\mu) \right\rangle.$$

Many examples of computations with the λ-ring formalism can be found in [23]. For other properties of inner plethysm, and applications to physics, the reader is referred to [1, 2, 15].

3. THE ADJOINTS OF THE ADAMS OPERATORS OF INNER PLETHYSM

For $d \in \mathbf{N}^*$ let

$$l_d := \frac{1}{d} \sum_{t \mid d} \mu_z(t)\, p_t^{d/t},$$

μ_z being the number-theoretic Moebius function. We note a representation theoretical interpretation of these polynomials.

3.2. THEOREM [5]. *Denote by $C_n := \langle (12 \cdots n) \rangle \subseteq S_n$ a transitive cyclic subgroup of order n in S_n. If $\chi : C_n \to \mathbf{C}^*$ is a faithful irreducible character, then the Frobenius characteristic of the induced character* $\mathrm{ind}_{C_n}^{S_n} \chi$ *is equal to l_n.*

In general, if the character χ is not faithful, the so-called von Sterneck function replaces the Moebius function [5]. If χ is trivial, then Euler's totient function comes into play. These polynomials appear in different contexts. They describe, for example, the action of the symmetric group on the free Lie algebra [17]. A combinatorial rule for their expansion in the basis of Schur functions has been given by W. Kraskiewicz and J. Weyman [22].

We are now ready to give a relation between the adjoints of Adams operators of both kinds of plethysms.

3.3. THEOREM. *Let $m \in \mathbf{N}^*$ and let $d_1, ..., d_r$ be all divisors of m. Then*

$$\hat{\phi}_m = M_r \circ R(l_{d_1} \otimes \cdots \otimes l_{d_r}) \circ (\phi_{d_1} \otimes \cdots \otimes \phi_{d_r}) \circ \Delta^r.$$

Proof. As we are dealing with homomorphisms with respect to ordinary multiplication, it suffices to show that the mappings on the left- and right-hand sides of the equation coincide on a generating set of $\mathbf{Q} \otimes \mathrm{Sym}(X)$. We will take the power sums p_n, $n \in \mathbf{N}^*$. Obviously, we have

$$\Delta^r p_n = \sum_{i=1}^{r} 1 \otimes \cdots \otimes p_n \otimes \cdots \otimes 1$$

and

$$\phi_d p_n = \begin{cases} d \cdot p_{n/d} & \text{if } \ d \mid n; \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$(\phi_d p_n)(l_d) = \begin{cases} \sum_{t \mid d} \mu_z(t)\, p_{tn/d}^{d/t} = \sum_{t \mid d} \mu_z(d/t)\, p_{n/t}^{t} & \text{if } \ d \mid n; \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Therefore the image of p_n under the right hand side is equal to

$$\sum_{d \mid m, d \mid n} \sum_{t \mid d} \mu_z(d/t)\, p_{n/t}^{t} = \sum_{d \mid (m,n)} \sum_{t \mid d} \mu_z(d/t)\, p_{n/t}^{t}$$
$$= \sum_{t \mid (m,n)} \Bigg(\sum_{t \mid d \mid (m,n)} \mu_z(d/t) \Bigg) p_{n/t}^{t}.$$

We can rewrite the inner sum as

$$\sum_{u \mid (m,n)/t} \mu_z(u) = \begin{cases} 1 & \text{if } \ t = (m, n) \\ 0 & \text{otherwise.} \end{cases}$$

Hence the outer sum breaks down to one single term and we get $p_{n/(m,n)}^{(m,n)}$ which is $\hat{\phi}_m(p_n)$ as desired. ∎

If we apply the theorem to $s_{(n)}$ or to get a compact notation to σ_1, then we recover a result which was proven directly and with different methods in [31, 38].

3.5. COROLLARY.

$$\hat{\phi}_m(\sigma_1)=\sigma_1\left(\sum_{d\mid n} l_d\right).$$

As the homogeneous component of degree l is the Frobenius characteristic of the mth-root number function of S_l, the equation in particular shows that the root number functions of the symmetric groups are proper characters. (This was conjectured by A. Kerber [14]).

Next, consider for $F\in \mathrm{Sym}(X)$

$$\hat{\phi}_m(\langle F\rangle)=\hat{\phi}_m(\sigma_1\cdot D_{\lambda_{-1}}F)=\hat{\phi}_m(\sigma_1)\cdot\hat{\phi}(D_{\lambda_{-1}}F)$$

which is, by Theorem 3.3 and Corollary 3.5,

$$\sigma_1\left(\sum_{d\mid m,d>1} l_d\right)\cdot\sigma_1\cdot M_r\circ R(l_{d_1}\otimes\cdots\otimes l_{d_r})\circ(\phi_{d_1}\otimes\cdots\otimes\phi_{d_r})\circ\Delta^r\circ D_{\lambda_{-1}}F.$$

Without loss of generality we may set $d_1:=1$. Using the associativity of M and coassociativity of Δ and taking into account that $R(l_1)=I=\phi_1$ we may rewrite this as

$$\sigma_1\left(\sum_{d\mid m,d>1} l_d\right)\cdot M\circ(\Gamma(1)$$
$$\otimes[M_{r-1}\circ R(l_{d_2}\otimes\cdots\otimes l_{d_r})\circ(\phi_{d_2}\otimes\cdots\otimes\phi_{d_r})\circ\Delta^{r-1})]\circ\Delta F.$$

If $F=s_{(n)}$, then this expression reads as

$$\sigma_1\left(\sum_{d\mid m,d>1} l_d\right)\cdot\sum_{\substack{a_1,\ldots,a_r\geqslant 0,\\ d_1a_1+\cdots+d_ra_r=n}}\langle s_{(a_1)}\rangle\, s_{(a_2)}(l_{d_2})\cdots s_{(a_r)}(l_{d_r}).$$

Because of Theorem 3.2 the plethysms correspond to proper characters. But the homogeneous component of degree $l+a_1$ in the series $\langle s_{(a_1)}\rangle$ corresponds to a proper character (i.e., $s_{(l,a_1)}$) as soon as $l\geqslant a_1$. This proves

3.6. COROLLARY. *For each partition α of n with at most two parts the image $\hat{\phi}_m s_\alpha$ is Frobenius characteristic of a proper character of S_n.*

For three-rowed partitions this is not true. $\hat{\phi}_2 s_{(1,1,1)}=-s_{(2,1)}$ provides a counterexample.

3.7. EXAMPLE. It is obvious that, if m is a multiple of $|S_l|$, the mth root number function of S_l is the character of the regular representation of S_l. But Corollary 3.5 shows that $\sigma_1(\sum_{d\geqslant 1} l_d)$ is the generating function for such root functions. As the generating function for the regular characters is $(1-p_1)^{-1}$, we get the well-known identity

$$\sigma_1\left(\sum_{d\geqslant 1} l_d\right)=\frac{1}{1-p_1}.$$

Let $a\in\mathbf{N}^{(\mathbf{N}^*)}$ be an infinite sequence with only finitely many entries greater than zero and take $k\in\mathbf{N}^*$ with $a_i=0$ for all $i>k$ (any such k will do). We define, viewing a as cycle type (of $\sum_i ia_i$), the canonical projection

$$\pi_{[a]}:\mathrm{Sym}(X)^{\otimes k}\to\bigotimes_{i=1}^{k}\mathrm{Sym}^{ia_i}(X)\subseteq\mathrm{Sym}(X)^{\otimes k}$$

and by this a homomorphism of vector spaces

$$\hat{\phi}_{[a]}=M_k\circ R(l_1\otimes\cdots\otimes l_k)\circ(\phi_1\otimes\cdots\otimes\phi_k)\circ\pi_{[a]}\circ\Delta^k. \tag{3.8}$$

It is easy to see that by Theorem 3.3

$$\hat{\phi}_m=\sum_{a:a_i\neq 0\Rightarrow i\mid m}\hat{\phi}_{[a]}. \tag{3.9}$$

The operators have a certain reciprocity property, a fact that has been noticed by several persons for the special case $F=s_{(n)}$, e.g., by Knörr [19] and Gessel [34].

3.10. THEOREM. *For all partitions α,β with cycle types $a,b\in\mathbf{N}^{(\mathbf{N}^*)}$, respectively, and all $F\in\mathrm{Sym}(X)$ we have*

$$(\hat{\phi}_{[a]}F,p_\beta^*)=(\hat{\phi}_{[b]}F,p_\alpha^*).$$

Here $p_\alpha^*:=p_\alpha/z_\alpha$.

Proof. (i) We first prove the equality for $F:=p_n$. Obviously, $\hat{\phi}_{[a]}p_n$ is nonzero only if $\alpha=(d^{n/d})$. In this case we get (cf. (3.4))

$$\hat{\phi}_{[a]}p_n=\sum_{(n/d)\mid t\mid n}\mu_z\left(\frac{td}{n}\right)p_t^{n/t},$$

so that we obtain

$$(\hat{\phi}_{[a]}p_n,p_\beta^*)=\begin{cases}\mu_z(dt/n) & \text{if } \alpha=(d^{n/d}),\ \beta=(t^{n/t}),\ (n/d)\mid t\mid n;\\ 0 & \text{else.}\end{cases}$$

But this equation is symmetric in α and β, as desired.

(ii) Let $F=GH$ and assume that the theorem is valid for G and H. Then, as $\sum_c \hat{\phi}_{[c]}$ is multiplicative and the sum is direct, we get

$$\hat{\phi}_{[a]}F=\sum_{r+s=a}\hat{\phi}_{[r]}(G)\,\hat{\phi}_{[s]}(H),$$

hence

$$\begin{aligned}(\hat{\phi}_{[a]}F, p^*_\beta) &= \sum_{r+s=a}(\hat{\phi}_{[r]}(G)\,\hat{\phi}_{[s]}(H), p^*_\beta)\\ &= \sum_{r+s=a}(\hat{\phi}_{[r]}(G)\otimes\hat{\phi}_{[s]}(H), \Delta p^*_\beta)\\ &= \sum_{r+s=a,\,\rho'\cup\sigma'=\beta}(\hat{\phi}_{[r]}(G)\otimes\hat{\phi}_{[s]}(H), p^*_{\rho'}\otimes p^*_{\sigma'})\\ &= \sum_{r+s=a,\,\rho'\cup\sigma'=\beta}(\hat{\phi}_{[r]}(G), p^*_{\rho'})(\hat{\phi}_{[s]}(H), p^*_{\sigma'}).\end{aligned}$$

By our assumption on G and H we may exchange r and s by the corresponding partitions ρ, σ and ρ', σ' by their cycle types r', s'. This gives

$$\sum_{r'+s'=b,\,\rho\cup\sigma=\alpha}(\hat{\phi}_{[r']}(G), p^*_\rho)(\hat{\phi}_{[s']}(H), p^*_\sigma)=(\hat{\phi}_{[b]}F, p^*_\alpha).$$

(iii) Starting with (i) the theorem follows with (ii) by induction. ∎

3.11. *Remark.* In the course of the proof we did not use the fact that μ_z is the Moebius function. In fact, let $f:\mathbf{N}^*\to\mathbf{C}$ be arbitrary and set $l^f_d := (1/d)\sum_{t\mid d}f(t)\,p_t^{d/t}$, $d>1$. Define $\hat{\phi}^f_{[a]}$ analogously to $\hat{\phi}_{[a]}$ with the only difference that l^f_d replaces l_d. Let Y be a countably infinite set of variables independent from X. Define, for $F\in \operatorname{Sym}(X)$,

$$P^f_F(X, Y):=\sum_\alpha(\hat{\phi}^f_{[a(\alpha)]}F)(X)\,p_\alpha(Y).$$

Then the above proof gives

$$P^f_F(X, Y)=P^f_F(Y, X). \tag{3.12}$$

3.13. *Remark.* It is worth noting that the cyclotomic identity can be derived from Theorem 3.10. Moreover, we recover certain generalizations.

Let u, v be two independent indeterminates and $F=s_{(n)}$. Then the substitution $p_i(X)\to u$, $p_j(Y)\to v$ gives the symmetric version of V. Strehl [35] (who, in fact, gives a combinatorial interpretation of this identity)

$$\prod_j\left(\frac{1}{1-ut^i}\right)^{l_i(v)}=\prod_j\left(\frac{1}{1-vt^i}\right)^{l_i(u)},$$

where $l_i(u)$, $l_i(v)$ denote the polynomials one gets by substituting u respectively v for each p_j in l_i.

If we only specialize $p_i(Y) \to 1$, then we get once more the identity (cf. Example 3.7)

$$\sigma_1\left(\sum_{d \geqslant 1} l_d\right) = \frac{1}{1-p_1}.$$

On the other hand, if we set $F := (-1)^n e_n$, we just get the inverse identities.

If we use the l_d^{φ}, φ the Euler totient function, as indicated in the corollary above, we obtain, for example,

$$\sigma_1\left(\sum_{d \geqslant 1} l_d^{\varphi}\right) = \prod_{i \geqslant 1} \frac{1}{1-p_i} = \sum_{\alpha} s_\alpha * s_\alpha.$$

This has also a representation theoretical interpretation. Consider the character κ_n of the permutation representation of S_n given by $(\pi, \sigma) \mapsto \pi\sigma\pi^{-1}$. The value of κ_n on the conjugacy class of type α is just z_α. So the right hand side of the identity is the generating series for the κ_n. On the other hand the permutation representation is the direct sum of the permutation representation on the orbits, i.e., the conjugacy classes of S_n. But each of these subrepresentations are induced from the trivial representation of the centralizer of an (arbitrary but fixed) element in the respective class. But the Frobenius characteristic of their characters is just a product of plethysms $s_{(a_1)}(l_1^{\varphi}) \cdots s_{(a_n)}(l_n^{\varphi})$. This clearly corresponds to the left hand side (cf. [12]).

3.14. *Remark.* As is well known, the Moebius and Euler functions are particular cases of the Von Sterneck functions (or Ramanujan sums) which can be defined by

$$\tau_k(n) = \sum_{(r,n)=1} \omega^{kr} \qquad (\omega = e^{2i\pi/n}).$$

One can thus wonder whether the above identities can be extended to the symmetric functions

$$l_n^{(k)} := \frac{1}{n} \sum_{d \mid n} \tau_k(d)\, p_d^{n/d},$$

i.e., $l_n^{(k)} = l_n^{\tau_k}$. But it is easy to see (by taking logarithms and rearranging the sum, but it is also a consequence of Remark 3.11) that, for a general f,

$$\sigma_1\left(\sum_{n \geqslant 1} l_n^f\right) = \prod_{m \geqslant 1} \left(\frac{1}{1-p_m}\right)^{(1/m)\sum_{d \mid m} f(d)},$$

and for $f=\tau_k$, the Von Sterneck theorem ($\sum_{d|m} \tau_k(d)=m$ if $m|k$ and $=0$ otherwise, see, e.g., [29]), yields the general identity

$$\sigma_1\left(\sum_{n\geqslant 1} l_n^{(k)}\right)=\prod_{d|k}\frac{1}{1-p_d}.$$

Since the $l_n^{(k)}$ are Frobenius images of proper characters, we see that the right-hand side is a positive sum of Schur functions.

3.15. Example. Let us denote, for each partition α of n with cycle type a,

$$P_\alpha:=\hat{\phi}_{[a]}s_{(n)}=s_{(a_1)}(l_1)\cdots s_{(a_n)}(l_n).$$

Then we may write

$$\sum_{\alpha\vdash n} P_\alpha(X)\,p_\alpha(Y)=\sum_{\beta,\gamma\vdash n} c_{\beta,\gamma}m_\beta(X)\,s_\gamma(Y).$$

Recently R. P. Stanley raised the question whether $c_{\beta,\gamma}\in\mathbf{N}$ [34]. That this is true is an immediate consequence of Remark 3.11, for we may exchange X and Y on the left-hand side. As $P_\alpha(Y)$ is the Frobenius image of a proper character, it decomposes into a linear combination of Schur functions with non-negative integral coefficients. And it is obvious that $p_\alpha(X)$ can be written in the basis of monomial symmetric functions with non-negative integral coefficients.

In [13] Kerber, Sänger, and Wagner gave a description of $\phi_j s_{\alpha/\beta}$. This leads to a formula for computing $\hat{\phi}_{[a]}s_{\alpha/\beta}$.

In the following we will describe, without proofs, their constructions. The central concept will be that of an r-quotient.

Let $r, s\in\mathbf{N}^*$. An (r, s)-configuration (or "abacus," cf. [10]) is a matrix $A\in\{0,1\}^{\{1,\ldots,r\}\times\mathbf{N}^*}$ with "1" appearing exactly s times. We will number the places in A by natural numbers, that is, by reading the matrix column by column from left to right and starting to count with zero.

It is possible to encode partitions by such configurations. To be precise, fix $s\in\mathbf{N}^*$ and let α be a partition with a most s parts. Then we assign to (α, s) the (r, s)-configuration which has a "1" only at those places with numbers

$$(\alpha_s, \alpha_{s-1}+1, \ldots, \alpha_1+s-1).$$

This mapping is a bijection between all partitions with at most s parts and all (r, s)-configurations. The inverse map can be described as follows. First one has to write the numbers of the 1's in an (r, s)-configuration in ascending order. Subtraction of the "staircase" $(0, 1, \ldots, s-1)$ yields the desired partition in reverse order.

Moreover, this gives a bijection between all partitions (to encode, let s be the length of the partition) and all configurations that have a "0" at place number 0.

3.16. EXAMPLE. Let $\alpha := (6, 5, 2, 1)$. To get the corresponding $(3, 5)$-configuration, we have to put 1's at $(0+0,\ 1+1,\ 2+2,\ 5+3,\ 6+4) = (0, 2, 4, 8, 10)$, thus

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \end{pmatrix}.$$

Let α be a partion with at most s parts and A its (r, s)-configuration. Each row of A itself can be viewed as an $(1, \cdot)$-configuration and the r-tuple of the corresponding partitions

$$(\alpha^{(1)}, ..., \alpha^{(r)})$$

is—up to cyclic permutation—defined by α. We call it the *r-quotient* of α.

3.17. EXAMPLE. In the above example we have to decode the three configurations

$$(1\,0\,0\,0\,0\cdots),\quad (0\,1\,0\,1\,0\cdots),\quad (1\,0\,1\,0\,0\cdots)$$

so that we obtain $((\ ), (2, 1), (1))$ as the 3-quotient of $\alpha := (6, 5, 2, 1)$. Here $(\ )$ is the unique partition of 0.

If β is another partition with at most s-parts, we may also form its (r, s)-configuration Z. Let $(\beta^{(1)}, ..., \beta^{(r)})$ be its r-quotient. Consider the following operations on configurations. Exchange a "1" and a "0" in the same row only if "0" if left to "1" and no 1's are between "0" and "1." If successive application of these operations transform A to Z, then we call the tuple (s is the same for both α and β)

$$(\alpha^{(1)}/\beta^{(1)}, ..., \alpha^{(r)}/\beta^{(r)})$$

the *r-quotient* of α/β. We define $s(\alpha, r, \beta)$ to be the product of the Schur functions corresponding to the skew partitions in the r-quotient. By applying the operations to A the order of the 1's may change. So transforming A to Z gives a permutation whose sign, as it only depends on α and β, we will denote by $\operatorname{sgn}(\alpha, r, \beta)$.

For convenience we define for all other pairs of partitions $s(\alpha, r, \beta) := 0 =: \operatorname{sgn}(\alpha, r, \beta)$. It should be noted that this construction can be interpreted as a generalization of the Murnaghan–Nakayama formula for skew characters. Getting Z from A is equivalent to removing admissible r-hooks from the diagram of α to obtain β; and the sign change is given by the leg-lengths of these hooks.

3.18. EXAMPLE. To continue the above example, consider

$$Z_1 := \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \cdots \end{pmatrix}, \qquad Z_2 := \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \end{pmatrix}, \qquad Z_3 := \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \end{pmatrix},$$

To find the corresponding partitions, we have to write down the numbers of the places occupied by 1's (recall that $s = 5$)

$$(0, 1, 2, 5, 7), \quad (0, 1, 2, 4, 5), \quad (0, 1, 5, 7, 8).$$

If we subtract the "staircase" $(0, 1, 2, 3, 4)$, we recover the partions in reverse order

$$(0-0, 1-1, 2-2, 5-3, 7-4) = (0, 0, 0, 2, 3),$$
$$(0-0, 1-1, 2-2, 4-3, 5-4) = (0, 0, 0, 1, 1),$$
$$(0-0, 1-1, 5-2, 7-3, 8-4) = (0, 0, 3, 4, 4).$$

Hence $(3, 2)$ corresponds to Z_1, $(1, 1)$ to Z_2, and $(4, 4, 3)$ to Z_3. Obviously, only Z_1 and Z_2 can be obtained from A, but not Z_3. Moreover, Z_2 is the unique configuration that can be obtained from A to which no transformation can be applied. The 3-quotient of $(3, 2)$ is (for $s = 5$), $((\ ), (1), (\ ))$ and that of $(1, 1)$ is $((\ ), (\ ), (\ ))$. Hence

$$s(\alpha, 3, (3, 2)) = s_{(2,1)/(1)} s_{(1)}, \qquad s(\alpha, 3, (1, 1)) = s_{(2,1)} s_{(1)},$$
$$s(\alpha, 3, (4, 4, 3)) = 0.$$

To get the corresponding sign changes, we number the 1's in A consecutively in the order given above. If we apply our transformations the A-numbered configurations are

$$\tilde{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 5 & \cdots \\ 2 & 0 & 4 & 0 & \cdots \end{pmatrix},$$

$$\tilde{Z}_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 5 & 0 & \cdots \\ 2 & 4 & 0 & 0 & \cdots \end{pmatrix}, \qquad \tilde{Z}_2 := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 5 & 0 & 0 & \cdots \\ 2 & 4 & 0 & 0 & \cdots \end{pmatrix},$$

so that the two permutations (in list notation) are [1, 3, 2, 4, 5], [1, 3, 2, 5, 4]. Hence

$$\operatorname{sgn}(\alpha, 3, (3, 2)) = \operatorname{sgn}([1, 3, 2, 4, 5]) = -1,$$

$$\operatorname{sgn}(\alpha, 3, (1, 1)) = \operatorname{sgn}([1, 3, 2, 5, 4]) = 1,$$

$$\operatorname{sgn}(\alpha, 3, (3, 2)) = 0.$$

This notation enables us to state

3.19. THEOREM [13]. $\phi_r s_{\alpha/\beta} = \operatorname{sgn}(\alpha, r, \beta)\, s(\alpha, r, \beta)$.

From this it is easy to get

3.20. THEOREM. *Let* $a \in \mathbf{N}^{(\mathbf{N}^*)}$ *and* $s \in \mathbf{N}^*$ *such that* $a_i = 0$ *for all* $i > s$. *Then*

$$\hat{\phi}_{[a]} s_{\alpha/\beta} = \sum \left(\prod_j \operatorname{sgn}(\gamma_{[j]}, j, \gamma_{[j-1]}) \cdot s(\gamma_{[j]}^{(i)}, j, \gamma_{[j-1]}^{(i)})(l_j) \right),$$

where the sum is over all chains of partitions

$$\beta = \gamma_{[0]} \subseteq \cdots \subseteq \gamma_{[s]} = \alpha,$$

with $|\gamma_{[j]}/\gamma_{[j-1]}| = ja_j$.

Combining this formula with (3.9), we obtain an algorithm for computing $\hat{\phi}_m(s_{\alpha/\beta})$.

3.21. EXAMPLE. Obviously,

$$\phi_3 s_{(6,5,2,1)} = 0,$$

$$\phi_3 s_{(6,5,2,1)/(1,1)} = s_{(2,1)} s_{(1)},$$

$$\phi_3 s_{(6,5,2,1)/(3,2)} = -s_{(2,1)/(1)} s_{(1)}.$$

It is easy to see that for $a := (2, 0, 4, 0, \ldots)$ there is only one relevant chain (i.e., equal terms are omitted), $(\ ) \subseteq (1^2) \subseteq (6, 5, 2, 1)$, from which it follows that

$$\hat{\phi}_{[a]} s_{(6,5,2,1)} = s_{(1^2)} \cdot (s_{(2,1)} s_1)(l_3).$$

For $a := (5, 0, 3, 0, \ldots)$ there are two relevant chains

$$(\ ) \subseteq (3, 2) \subseteq (6, 5, 2, 1) \qquad \text{and} \qquad (\ ) \subseteq (4, 1) \subseteq (6, 5, 2, 1).$$

As $s((6, 5, 2, 1), 3, (4, 1)) = s_{(2,1)}$, $\operatorname{sgn}((6, 5, 2, 1), 3, (4, 1)) = 1$, we get

$$\begin{aligned}\hat{\phi}_{[a]} s_{(6,5,2,1)} &= -s_{(3,2)} \cdot (s_{(2,1)/(1)} s_1)(l_3) + s_{(4,1)} \cdot s_{(2,1)}(l_3)\\ &= -s_{(3,2)} \cdot (l_3)^3 + s_{(4,1)} \cdot s_{(2,1)}(l_3).\end{aligned}$$

To obtain $\hat{\phi}_3 s_{(6,5,2,1)}$ we have to consider three more terms.
For $a := (8, 0, 2, 0, ...)$ we have three relevant chains

$$(\ ) \subseteq (4, 4) \subseteq (6, 5, 2, 1),$$

$$(\ ) \subseteq (6, 2) \subseteq (6, 5, 2, 1),$$

$$(\ ) \subseteq (3, 2, 2, 1) \subseteq (6, 5, 2, 1),$$

whence

$$\begin{aligned}\hat{\phi}_{[a]} s_{(6,5,2,1)} &= s_{(4,4)} \cdot s_{(2,1)/(1)}(l_3) - s_{(6,2)} \cdot (s_{(2,1)/(2)} s_{(1)})(l_3)\\ &\quad + s_{(3,2,2,1)} \cdot (s_{(2,1)/(1^2)} s_{(1)})(l_3)\\ &= s_{(4,4)} \cdot (l_3)^2 - s_{(6,2)} \cdot (l_3)^2 + s_{(3,2,2,1)} \cdot (l_3)^2.\end{aligned}$$

For $\alpha := (11, 0, 1, 0, ...)$ there are again three relevant chains,

$$(\ ) \subseteq (4, 4, 2, 1) \subseteq (6, 5, 2, 1)$$

$$(\ ) \subseteq (6, 2, 2, 1) \subseteq (6, 5, 2, 1),$$

$$(\ ) \subseteq (6, 5) \subseteq (6, 5, 2, 1),$$

hence

$$\hat{\phi}_{[a]} s_{(6,5,2,1)} = -s_{(4,4,2,1)} l_3 + s_{(6,2,2,1)} l_3 - s_{(6,5)} l_3.$$

And finally there is $a := (14, 0, 0, 0, ...)$ where we get

$$\hat{\phi}_{[a]} s_{(6,5,2,1)} = s_{(6,5,2,1)}.$$

From this it is straightforward to give the decomposition of $\hat{\phi}_3 s_{(6,5,2,1)}$ into Schur functions, but the expression is too large to be given here.

4. Action of the Adams Operators on Stable Characters

The results of the preceding section allow us to derive in a systematic way the stable reductions of the series $\hat{\psi}_k(\langle s_\lambda \rangle)$. We give the details for $k = 2$ and for $k = 3$. It is possible to go further but the resulting formulas are too complicated to be really useful. In all cases, the substantial part of the computation is to extract a factor σ_1 from a series $\hat{\psi}_k(\sigma_1 F)$. It is then a simple matter to understand what happens when F is replaced by $D_{\lambda_{-1}}F$.

First we will restrict to the case that k is prime. We start dually using (3.3). Then

$$D_{\sigma_1} \circ \hat{\phi}_k = D_{\sigma_1} \circ M \circ R(l_1 \otimes l_k) \circ (\phi_1 \otimes \phi_k) \circ \Delta.$$

As $R(l_1) = I = \phi_1$ and D_{σ_1} is multiplicative we get

$$M \circ (D_{\sigma_1} \otimes D_{\sigma_1}) \circ (I \otimes R(l_k)) \circ (I \otimes \phi_k) \circ \Delta$$

hence, as $D_{\sigma_1} \circ R(G) = R(D_{\sigma_1} G)$,

$$M \circ (I \otimes R(D_{\sigma_1} l_k)) \circ (I \otimes \phi_k) \circ (D_{\sigma_1} \otimes I) \circ \Delta.$$

Now, $D_{\sigma_1}(l_k(X)) = l_k(X+1) = l_k + \tilde{l}_k$ with $\tilde{l}_k = (1/k) \sum_{i=1}^{k-1} \binom{k}{i} l_1^i$. Furthermore it is easy to see that $(D_{\sigma_1} \otimes I) \circ \Delta = \Delta \circ D_{\sigma_1}$. Hence we can write

$$\begin{aligned} &M \circ (I \otimes R(l_k + \tilde{l}_k)) \circ (I \otimes \phi_k) \circ \Delta \circ D_{\sigma_1} \\ &\quad = M \circ (I \otimes M) \circ (I \otimes R(l_k) \otimes R(\tilde{l}_k)) \circ (I \otimes \Delta) \circ (I \otimes \phi_k) \circ \Delta \circ D_{\sigma_1}. \end{aligned}$$

As ψ_k is multiplicative and $R(l_1) = I = \phi_1$, we get

$$M \circ (I \otimes M) \circ (R(l_1) \otimes R(l_k) \otimes R(\tilde{l}_k)) \circ (\phi_1 \otimes \phi_k \otimes \phi_k) \circ (I \otimes \Delta) \circ \Delta \circ D_{\sigma_1},$$

and, by associativity of M and coassociativity of Δ,

$$M \circ (M \otimes I) \circ (R(l_1) \otimes R(l_k) \otimes R(\tilde{l}_k)) \circ (\phi_1 \otimes \phi_k \otimes \phi_k) \circ (\Delta \otimes I) \circ \Delta \circ D_{\sigma_1}.$$

Using Theorem 3.3 again we finally arrive at

4.1. THEOREM. $D_{\sigma_1} \circ \hat{\phi}_k = M \circ (\hat{\phi}_k \otimes (R(\tilde{l}_k) \circ \phi_k)) \circ \Delta \circ D_{\sigma_1}$.

In the case $k = 2$ we immediately have (as $\tilde{l}_2 = l_1$)

4.2. THEOREM. $D_{\sigma_1} \circ \hat{\phi}_2 = M \circ (\hat{\phi}_2 \otimes \phi_2) \circ \Delta \circ D_{\sigma_1}$, *and dually, for any symmetric function F,*

$$\hat{\psi}_2(\sigma_1 F) = \sigma_1 \cdot M \circ (\hat{\psi}_2 \otimes \psi_2) \circ \Delta F.$$

For $k = 3$ we observe that

$$R(\tilde{l}_3) = R(l_1 + l_1^2) = M \circ (R(l_1) \otimes R(l_1^2)) \circ \Delta$$

which is

$$M\circ(I\otimes M)\circ(I\otimes R(l_1)\otimes R(l_1))\circ(I\otimes\delta)\circ\Delta=M\circ(I\otimes M)\circ(I\otimes\delta)\circ\Delta.$$

Hence

$$R(\tilde{l}_3)\circ\phi_3=M_3\circ(\phi_3\otimes(\delta\circ\phi_3))\circ\Delta$$

and Theorem 4.1 gives

4.3. THEOREM. $\hat{\psi}_3(\sigma_1 F)=\sigma_1\cdot M_3\circ(\hat{\psi}_3\otimes\psi_3\otimes(\psi_3\circ m))\circ\Delta^4F.$

Now, to compute the image of a stable character $\langle F\rangle$, we start with

$$\begin{aligned}\hat{\psi}_2(\langle F\rangle)&=\hat{\psi}_2(\sigma_1 D_{\lambda_{-1}}F)=\sigma_1\cdot M\circ(\hat{\psi}_2\otimes\psi_2)\circ\Delta\circ D_{\lambda_{-1}}F\\&=\sigma_1\cdot D_{\lambda_{-1}}\circ(D_{\sigma_1}\circ M\circ(\hat{\psi}_2\otimes\psi_2)\circ\Delta\circ D_{\lambda_{-1}}F)\end{aligned}$$

using Theorem 4.2 and the fact that $D_{\lambda_{-1}}\circ D_{\sigma_1}=I$. As D_{σ_1} is multiplicative we may write this as

$$\begin{aligned}&\langle D_{\sigma_1}\circ M\circ(\hat{\psi}_2\otimes\psi_2)\circ\Delta\circ D_{\lambda_{-1}}F\rangle\\&\quad=\langle M\circ(D_{\sigma_1}\otimes D_{\sigma_1})\circ(\hat{\psi}_2\otimes\psi_2)\circ\Delta\circ D_{\lambda_{-1}}F\rangle.\end{aligned}$$

This again is equal to

$$\langle M\circ(\hat{\psi}_2\otimes\psi_2)\circ\Delta\circ D_{\sigma_1(l_1+l_2)}F\rangle,$$

for $D_{\sigma_1}\circ\psi_2=\psi_2\circ D_{\sigma_1}$, $D_{\sigma_1}\circ\hat{\psi}_2=\hat{\psi}_2\circ D_{\hat{\phi}_2(\sigma_1)}=\hat{\psi}_2\circ D_{\sigma_1(l_1+l_2)}$, and $\Delta\circ D_{\sigma_1}=(D_{\sigma_1}\otimes I)\circ\Delta$. Since $l_1=e_1$, $l_2=e_2$, we finally obtain

4.4. THEOREM. $\hat{\psi}_2(\langle F\rangle)=\langle M\circ(\hat{\psi}_2\otimes\psi_2)\circ\Delta\circ D_{\sigma_1(e_1+e_2)}F\rangle.$

By a well known identity due to Schur [27, I.5, Examples 4, 10] [24], we have

$$\sigma_1(e_1+e_2)=\prod_i\frac{1}{1-x_i}\prod_{i<j}\frac{1}{1-x_ix_j}=\sum_\alpha s_\alpha$$

so that

$$\hat{\psi}_2(\langle s_\lambda\rangle)=\left\langle\sum_{\alpha,\beta}\psi_2(D_{s_\alpha}D_{s_\beta}s_\lambda)\,\hat{\psi}_2(s_\beta)\right\rangle. \tag{4.5}$$

Taking into account the facts that

$$\hat{s}_{(2)}(F)=(F*F+\hat{\psi}_2(F))/2 \qquad\text{and}\qquad \hat{s}_{(11)}(F)=(F*F-\hat{\psi}_2(F))/2,$$

and using Littlewood's formula 2.3 for the expansion of $\langle s_\lambda\rangle * \langle s_\lambda\rangle$, we recover exactly Theorem X of [26].

By a similar calculation it is also possible to describe $\hat{\psi}_3(\langle F\rangle)$. Indeed, it follows from Theorem 4.3 that

$$\begin{aligned}
\hat{\psi}_3(\langle F\rangle) &= \hat{\psi}_3(\sigma_1\, D_{\lambda_{-1}} F)\\
&= \sigma_1 \cdot D_{\lambda_{-1}} \circ D_{\sigma_1} \circ M_3 \circ (\hat{\psi}_3 \otimes \psi_3 \otimes (\psi_3 \circ m)) \circ \Delta^4 \circ D_{\lambda_{-1}} F\\
&= \langle M_3 \circ ((D_{\sigma_1} \circ \hat{\psi}_3) \otimes (D_{\sigma_1} \circ \psi_3) \otimes (D_{\sigma_1} \circ \psi_3 \circ m)) \circ \Delta^4 \circ D_{\lambda_{-1}} F\rangle\\
&= \langle M_3 \circ ((\hat{\psi}_3 \circ D_{\sigma_1(l_1+l_3)}) \otimes \psi_3 \otimes (\psi_3 \circ D_{\sigma_1} \circ m)) \circ \Delta^4 F\rangle\\
&= \langle M_3 \circ (\hat{\psi}_3 \otimes \psi_3 \otimes (\psi_3 \circ D_{\sigma_1} \circ m)) \circ \Delta^4 \circ D_{\sigma_1(l_1+l_3)} F\rangle.
\end{aligned}$$

Expanding the coproduct by means of Schur functions gives

$$\begin{aligned}
\hat{\psi}_3(\langle F\rangle) &= \left\langle \sum_\alpha M_3 \circ (\hat{\psi}_3 \otimes \psi_3 \otimes (\psi_3 \circ D_{\sigma_1} \circ m)) \circ (D_{s_\alpha} \circ D_{\sigma_1(l_1+l_3)} F) \otimes \Delta^3 s_\alpha)\right\rangle\\
&= \left\langle \sum_{\alpha\beta\gamma} (\hat{\psi}_3 \circ D_{s_\alpha} \circ D_{\sigma_1(l_1+l_3)} F)(\psi_3 \circ D_{s_\gamma} \circ D_{s_\beta} s_\alpha)(\psi_3 \circ D_{\sigma_1} s_\beta * s_\gamma)\right\rangle.
\end{aligned}$$

Since $D_{\sigma_1}(G * H) = \sum_\theta D_{s_\theta} G * D_{s_\theta} H$, we finally get

4.6. THEOREM. *For any symmetric function F the following holds*

$$\hat{\psi}_3(\langle F\rangle) = \left\langle \sum_{\alpha\beta\gamma\theta} (\hat{\psi}_3 \circ D_{s_\alpha} \circ D_{\sigma_1(l_1+l_3)} F) \cdot (\psi_3 \circ D_{s_\gamma} \circ D_{s_\beta} s_\alpha) \cdot \psi_3(s_{\beta/\theta} * s_{\gamma/\theta})\right\rangle.$$

4.7. EXAMPLE. Let us conclude with some examples for $\hat{\phi}_2(\langle F\rangle)$, namely $F = s_{(n)}$ and $F := e_n$.

The following is immediate:

$$\hat{\psi}_2(s_{(n)}) = s_{(n)} = \hat{\psi}_2(e_n).$$

There is an involution on $\mathrm{Sym}(X)$ which maps $F(X) \to F(-X)$ and it is easy to see that it commutes with ψ_2 so that in particular

$$\psi_2(e_n(X)) = (-1)^n\, (\psi_2(s_{(n)}))(-X).$$

Formulas for decomposing $\psi_k(s_{(n)})$ are known. One possibility is to use k-quotients. We will do this for $k = 2$. To compute $\psi_2(s_{(n)})$ we describe, using the adjoint, those partitions α of $2n$ with $(\phi_2(s_\alpha), s_{(n)}) = (s(\alpha, 2, (\ )), s_{(n)}) \neq 0$. But to get this the 2-quotient of α must be of the form $(s_{(r)}, s_{(l)})$, $r+l=n$. As we may assume that the place 0 in the configuration contains zero, we see that the 0-row will contain exactly one entry "1" and the 1-row at most one entry "1." Hence α has at most two rows and it is easy to see that all partitions of $2n$ can be transformed to be the zero partition by removing two hooks. Moreover, the corresponding sign is one, if the first part is even and -1 if it is odd. Hence

$$\psi_2 s_{(n)} = \sum_{i=0}^{n} (-1)^i s_{(2n-i,i)}.$$

Now, to compute $\hat{\psi}_2(\langle s_r \rangle)$ by Theorem 4.4, we see that

$$D_{s_\gamma} s_{(r)} = \begin{cases} s_{(r-l)} & \text{if } \gamma = (l), l \leqslant r \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\hat{\psi}_2(\langle s_{(r)} \rangle) = \left\langle \sum_{l=0}^{r} \sum_{k=0}^{r-l} \sum_{i=0}^{r-l-k} (-1)^i s_{(2(r-l-k)-i,i)} s_k \right\rangle$$

and

$$\hat{\psi}_2(\langle e_r \rangle) = \left\langle \sum_{l=0}^{r} \sum_{k=0}^{r-l} \sum_{i=0}^{r-l-k} (-1)^{r-l-k-i} s_{(2^i, 1^{2(r-l-k-i)})} s_k \right\rangle.$$

Since the internal product of two hook (or two-rowed) Schur functions is known [30, 37], this gives the complete reduction of $\hat{h}_2(s_\lambda)$ and $\hat{e}_2(s_\lambda)$ for $\lambda = (n-k, k)$ or $\lambda = (n-k, 1^k)$.

4.8. Example. If we use Theorem 4.3 to compute $\hat{\psi}_3(\sigma_1 F)$ for $F := s_{(r)}$ and $F := e_r$, we get (as $\hat{\psi}_3(s_{(i)}) = s_{(i)}$ and $\hat{\psi}_3(e_i) = e_i$)

$$\begin{aligned} \hat{\psi}_3(\sigma_1 s_{(r)}) &= \sigma_1 \sum_{i+j+k+l=r} \hat{\psi}_3(s_{(i)})\, \psi_3 s_{(j)} \psi_3(s_{(k)} * s_{(l)}) \\ &= \sigma_1 \sum_{i+j+2k=r} s_{(i)}\, \psi_3 s_{(j)} \psi_3 s_{(k)}. \end{aligned}$$

Similarly we have

$$\hat{\psi}_3(\sigma_1 e_r) = \sigma_1 \sum_{i+j+2k=r} e_i \psi_3 e_j \psi_3 s_{(k)}.$$

And, for the stable characters, we get

$$\hat{\psi}_3(\langle s_{(r)}\rangle) = \left\langle \sum_{i+j+2k=r} D_{\sigma_1} s_{(i)} \psi_3 s_{(j)} \psi_3 s_{(k)} * \psi_3 D_{\sigma_1} s_{(k)} \right\rangle.$$

$$\hat{\psi}_3(\langle e_r\rangle) = \left\langle \sum_{i+j+2k=r} D_{\sigma_1} e_i \psi_3 e_j \psi_3 D_{\sigma_1} s_{(k)} \right\rangle.$$

5. Branching Rules and Reciprocity Formulas

Another problem investigated by Littlewood in [26] is the one of computing the stable reduction of the images by Schur functors of the fundamental representation $[n-1, 1]$ of S_n, corresponding to the generating series $\langle s_{(1)}\rangle = \sum_{n\geqslant 1} s_{(n-1,1)}$. It is simpler to study first the images of the series $\sigma_1 s_{(1)}$, corresponding to the representations of symmetric groups by permutation matrices. In other words, the resulting formula gives the branching rule $GL(n) \supset S_n$, S_n being embedded in $GL(n)$ as the group of permutation matrices. This rule can be stated in the form of a reciprocity formula:

5.1. Theorem. *For any F and $G \in \mathrm{Sym}(X)$ one has*

$$(\hat{F}(\sigma_1 s_{(1)}), G) = (F, G(\sigma_1)),$$

where $G(\sigma_1)$ is the outer plethysm of σ_1 by G.

First we need

5.2. Lemma. $\hat{\psi}_k(\sigma_1 s_{(1)}) = \sigma_1 \sum_{d\mid k} p_d$.

Proof. Let $p_\nu = p_1^{\gamma_1} \cdots p_n^{\gamma_n}$. Then,

$$(\hat{\psi}_k(\sigma_1 s_{(1)}), p_\nu) = (s_{(1)}, D_{\sigma_1} \hat{\phi}_k(p_\nu)) = \sum_{d\mid k} d\gamma_d$$

$$= \left(\sum_{d\mid k} p_d, (1+p_1)^{\gamma_1} \cdots (1+p_n)^{\gamma_n} \right)$$

$$= \left(\sum_{d\mid k} p_d, D_{\sigma_1} p_\nu \right). \quad \blacksquare$$

Proof of Theorem 5.1. Take $F = p_\lambda$ and $G = p_\mu$, with $\lambda = (\lambda_1, ..., \lambda_r) = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$ and $\mu = (\mu_1, ..., \mu_s) = (1^{\beta_1} 2^{\beta_2} \cdots n^{\beta_n})$. Then, on one hand,

$$\begin{aligned}
(\hat{p}_\lambda(\sigma_1 s_{(1)}), p_\mu) &= (\hat{\psi}_{\lambda_1}(\sigma_1 s_{(1)}) * \hat{\psi}_{\lambda_2}(\sigma_1 s_{(1)}) * \cdots * \hat{\psi}_{\lambda_r}(\sigma_1 s_{(1)}), p_\mu) \\
&= \left(\bigotimes_{h=1}^{r} \hat{\psi}_{\lambda_h}(\sigma_1 s_{(1)}), \delta^r p_\mu \right) \\
&= \left(\bigotimes_{h=1}^{r} \left(\sigma_1 \sum_{d_h \mid \lambda_h} p_{d_h} \right), p_\mu \otimes \cdots \otimes p_\mu \right) \\
&= \prod_{h=1}^{r} \left(\sum_{d_h \mid \lambda_h} p_{d_h}, \prod_{i \geqslant 1} (1+p_i)^{\beta_i} \right) \\
&= \prod_{h=1}^{r} \left(\sum_{d_h \mid \lambda_h} p_{d_h}, \sum_{j \geqslant 1} \beta_j p_j \right) \\
&= \prod_{i \geqslant 1} \left(\sum_{d \mid i} p_d, \sum_{j \geqslant 1} \beta_j p_j \right)^{\alpha_i} \\
&= \prod_{i \geqslant 1} \left[\sum_{d \mid i} \beta_d (p_d, p_d) \right]^{\alpha_i} \\
&= \prod_{i \geqslant 1} \left[\sum_{d \mid i} d\beta_d \right]^{\alpha_i},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
(p_\lambda, p_\mu(\sigma_1)) &= \left(\bigotimes_{h=1}^{r} p_{\lambda_h}, \Delta^r [p_\mu(\sigma_1)] \right) \\
&= \left(\bigotimes_{h=1}^{r} p_{\lambda_h}, p_\mu(\sigma_1) \otimes \cdots \otimes p_\mu(\sigma_1) \right) \\
&= \prod_{h=1}^{r} (p_{\lambda_h}, p_\mu(\sigma_1)),
\end{aligned}$$

and writing

$$\sigma_1 = 1 + \sum_{k \geqslant 1} \frac{p_k}{k} + O_2(p),$$

where $O_2(p)$ stands for terms with degree at least two in the p_i, this is

$$\begin{aligned}
\prod_{h=1}^{r} \left(p_{\lambda_h}, \prod_{j \geqslant 1} \left[1 + \sum_{k \geqslant 1} \frac{p_{kj}}{k} \right]^{\beta_j} \right) &= \prod_{h=1}^{r} \left(p_{\lambda_h}, \sum_{j,k \geqslant 1} \beta_j \frac{p_{kj}}{k} \right) \\
&= \prod_{i \geqslant 1} \left(p_i, \sum_{d \mid i} \frac{\beta_d}{(i/d)} p_i \right)^{\alpha_i} \\
&= \prod_{i \geqslant 1} \left[\sum_{d \mid i} d\beta_d \right]^{\alpha_i}. \quad \blacksquare
\end{aligned}$$

Next, we observe that

$$\hat{F}(\langle s_{(1)} \rangle) = \hat{F}(\sigma_1 s_{(1)} - \sigma_1) = (\widehat{D_{\lambda_{-1}} F})(\sigma_1 s_{(1)}).$$

Indeed, taking $F = s_\lambda$ we have

$$\hat{s}_\lambda(\sigma_1 s_{(1)} - \sigma_1) = \sum_\mu (-1)^{|\mu|} \hat{s}_{\lambda/\mu}(\sigma_1 s_{(1)}) * \hat{s}_{\mu'}(\sigma_1)$$

and $\hat{s}_{\mu'}(\sigma_1) \neq 0$ only when $l(\mu') = 1$, i.e., when $\mu = (1^r)$, so that

$$\hat{s}_\lambda(\langle s_{(1)} \rangle) = \sum_{r \geqslant 0} (-1)^r \hat{s}_{\lambda/1^r}(\sigma_1 s_{(1)}) = (\widehat{D_{\lambda_{-1}} s_\lambda})(\sigma_1 s_{(1)}).$$

Thus, we obtain

5.3. Corollary. $(\hat{F}(\langle s_{(1)} \rangle), G) = (D_{\lambda_{-1}} F, G(\sigma_1))$.

This corollary is in fact equivalent to Littlewood's branching rule for $GL(n-1) \supset S_n$ (Theorem XI of [26]). To see this, take an integer N greater then the weight of F. Then,

$$\begin{aligned}
&(D_{\lambda_{-1}} F, G(\sigma_1)) \\
&\quad = (D_{\lambda_{-1}(h_1 + \cdots + h_N)\sigma_1(h_2 + \cdots + h_N)} F, G(1 + h_1 + \cdots + h_N)) \\
&\quad = (D_{\sigma_1(h_2 + \cdots + h_N)} F, M_N \circ (I \otimes R(h_2) \otimes \cdots \otimes R(h_N)) \circ \Delta^N(\lambda_{-1} D_{\sigma_1} G)) \\
&\quad = \sum_{\substack{\lambda^{(2)}, \ldots, \lambda^{(N)} \\ r_2, \ldots, r_N}} (D_{h_{r_2}(h_2)} \cdots D_{h_{r_N}(h_N)} F, \\
&\qquad\qquad D_{s_{\lambda^{(2)}}} \cdots D_{s_{\lambda^{(N)}}}(\lambda_{-1} D_{\sigma_1} G)\, s_{\lambda^{(2)}}(h_2) \cdots s_{\lambda^{(N)}}(h_N)) \\
&\quad = \sum_{\substack{\lambda^{(2)}, \ldots, \lambda^{(N)} \\ r_2, \ldots, r_N}} (s_{\lambda^{(2)}} \cdots s_{\lambda^{(N)}} D_{s_{\lambda^{(2)}}(h_2)} \\
&\qquad\qquad \cdots D_{s_{\lambda^{(N)}}(h_N)} D_{h_{r_2}(h_2)} \cdots D_{h_{r_N}(h_N)} F, \lambda_{-1} D_{\sigma_1} G) \\
&\quad = \sum_{\substack{\lambda^{(2)}, \ldots, \lambda^{(N)} \\ r_2, \ldots, r_N}} (\langle s_{\lambda^{(2)}} \cdots s_{\lambda^{(N)}} D_{s_{\lambda^{(2)}}(h_2)} \\
&\qquad\qquad \cdots D_{s_{\lambda^{(N)}}(h_N)} D_{h_{r_2}(h_2)} \cdots D_{h_{r_N}(h_N)} F \rangle, G),
\end{aligned}$$

which is equivalent to Littlewood's expression.

6. Littlewood's Formulas for $O(n)$ and $Sp(n)$ Characters

For the sake of completeness, we show here how the formalism of the preceding sections can be used to prove Littlewood's formulas for products and plethysms of characters of orthogonal and sympletic groups. These formulas deal with the symmetric functions

$$o_\lambda = D_{\lambda_{-1}(h_2)} s_\lambda$$

and

$$sp_\lambda = D_{\lambda_{-1}(e_2)} s_\lambda$$

which have been called *universal characters* [20], since their specialization to appropriate alphabets gives the irreducible characters of $O(n)$ and $Sp(n)$ (see also [36] for a survey of this subject.)

Let us first examine the reduction of tensor products for $O(n)$. Writing

$$o_\lambda o_\mu = \sum_\nu c^\nu_{\lambda\mu} o_\nu = D_{\lambda_{-1}(h_2)} \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

we have to find for the sum $\sum_\nu c^\nu_{\lambda\mu} s_\nu$ some closed form which could be expanded by known operations on Schur functions. Now,

$$\begin{aligned}
(o_\lambda o_\mu, F) &= (o_\lambda \otimes o_\mu, \Delta F)\\
&= ((D_{\lambda_{-1}(h_2)} \otimes D_{\lambda_{-1}(h_2)})(s_\lambda \otimes s_\mu), \Delta F)\\
&= (s_\lambda \otimes s_\mu, \Delta[\lambda_{-1}(h_2)] \cdot \delta\sigma_1 \cdot \Delta F)\\
&\qquad [\text{since } \Delta[\lambda_{-1}(h_2)] = \lambda_{-1}(h_2) \otimes \lambda_{-1}(h_2) \cdot \delta\lambda_{-1}]\\
&= (D_{\delta\sigma_1} s_\lambda \otimes s_\mu, \Delta[\lambda_{-1}(h_2) F])\\
&= (D_{\lambda_{-1}(h_2)}[M \circ D_{\delta\sigma_1} s_\lambda \otimes s_\mu], F)
\end{aligned}$$

so that

6.1. Theorem [26, Theorem I]. $o_\lambda o_\mu = D_{\lambda_{-1}(h_2)}[M \circ D_{\delta\sigma_1} s_\lambda \otimes s_\mu] = D_{\lambda_{-1}(h_2)} \sum_\theta s_{\lambda/\theta} s_{\mu/\theta}$, *i.e.*, $\sum c^\nu_{\lambda\mu} s_\nu = \sum_\theta s_{\lambda/\theta} s_{\mu/\theta}$.

The case of $sp_\lambda sp_\mu$ is completely similar [26, Theorem II]. We note that our proof of Theorem 6.1 is essentially identical to the ones already given by several authors [21, 16, 4].

On the other hand, Littlewood's formulas for a plethysm with orthogonal and sympletic group characters seem to have been overlooked. These formulas give the reductions of the images of $O(n)$ and $Sp(n)$ representations by Schur functors of weight 2 and 3. To obtain them in the form given by Littlewood, it is clearly sufficient to determine the images by ψ_2 and ψ_3 of o_λ and sp_λ.

To begin with, let us consider the general cases $\psi_r(o_\lambda)$, $\psi_r(sp_\lambda)$. For orthogonal group characters we write it as

$$\psi_r(o_\lambda) = D_{\lambda_{-1}(h_2)}[D_{\sigma_1(h_2)}\psi_r(D_{\lambda_{-1}(h_2)}s_\lambda)]$$

and we transform the expression between brackets using duality:

$$\begin{aligned}(D_{\sigma_1(h_2)}\psi_r(D_{\lambda_{-1}(h_2)}s_\lambda), F) &= ((D_{\lambda_{-1}(h_2)}s_\lambda, \phi_r(\sigma_1(h_2))\,\phi_r(F))\\ &= (s_\lambda, \lambda_{-1}(h_2)\,\phi_r(\sigma_1(h_2))\,\phi_r(F)).\end{aligned}$$

The same can be done for symplectic characters. Thus we have to consider the expressions

$$\lambda_{-1}(h_2)\,\phi_r(\sigma_1(h_2)) \qquad \text{and} \qquad \lambda_{-1}(e_2)\,\phi_r(\sigma_1(e_2))$$

in the orthogonal and symplectic case, respectively.

Let now $r > 1$ be odd. Then

$$\phi_r\sigma_1(h_2) = \phi_r \exp\left[\sum_{k\geqslant 1}\frac{p_k^2}{2k} + \sum_{k\geqslant 1}\frac{p_{2k}}{2k}\right] = \exp\left[\sum_{k\geqslant 1}\frac{(\phi_r p_k)^2}{2k} + \sum_{k\geqslant 1}\frac{\phi_r p_{2k}}{2k}\right]$$

which is, as r is odd,

$$\begin{aligned}\exp\left[\sum_{k\geqslant 1, r\mid k}\frac{r^2p_{k/r}^2}{2k} + \sum_{k\geqslant 1, r\mid k}\frac{rp_{2k/r}}{2k}\right] &= \exp\left[\sum_{k\geqslant 1, r\mid k}\binom{r}{2}\frac{p_{k/r}^2}{k/r} + \sum_{k\geqslant 1, r\mid k}\frac{p_{2k/r}}{2k/r}\right]\\ &= \exp\left[\sum_{j\geqslant 1}\binom{r}{2}\frac{p_j^2}{j} + \sum_{j\geqslant 1}\frac{p_{2j}}{2j}\right];\end{aligned}$$

and multiplying this expression with

$$\lambda_{-1}(h_2) = \exp\left[-\sum_{j\geqslant 1}\frac{p_j^2 + p_{2j}}{2j}\right]$$

we find

$$\lambda_{-1}(h_2)\,\phi_r(\sigma_1(h_2)) = \exp\sum_{i\geqslant 1}\left(\frac{r-1}{2}\right)\frac{p_i^2}{i} = \sigma_1(X^2)^{(r-1)/2}.$$

Hence

$$\lambda_{-1}(h_2)\,\phi_r(\sigma_1(h_2)) = \left(\sum_\alpha s_\alpha^2\right)^{(r-1)/2}, \qquad r > 1 \text{ odd}, \tag{6.2}$$

and, analogously, for sympletic group characters

$$\lambda_{-1}(e_2)\,\phi_r(\sigma_1(e_2))=\left(\sum_\alpha s_\alpha^2\right)^{(r-1)/2},\qquad r>1 \text{ odd.} \tag{6.3}$$

For even r we find

$$\begin{aligned}
\phi_r\sigma_1(h_2)&=\exp\left[\sum_{k\geqslant 1,\,r\mid k}\frac{r^2p_{k/r}^2}{2k}+\sum_{k\geqslant 1,\,(r/2)\mid k}\frac{rp_{2k/r}}{2k}\right]\\
&=\exp\left[\sum_{k\geqslant 1,\,r\mid k}\binom{r}{2}\frac{p_{k/r}^2}{k/r}+\sum_{k\geqslant 1,\,(r/2)\mid k}\frac{p_{k/(r/2)}}{k/(r/2)}\right]\\
&=\exp\left[\sum_{j\geqslant 1}\binom{r}{2}\frac{p_j^2}{j}+\sum_j\frac{p_j}{j}\right]\\
&=\sigma_1(X^2)^{(r-2)/2}\cdot\sigma_1\cdot\exp\sum_{j\geqslant 1}\frac{p_j^2}{j}.
\end{aligned}$$

Thus

$$\lambda_{-1}(h_2)\,\phi_r(\sigma_1(h_2))=\left(\sum_\alpha s_\alpha^2\right)^{(r-2)/2}\sigma_1\sigma_1(e_2)=\left(\sum_\alpha s_\alpha^2\right)^{(r-2)/2}\sigma_1(e_1+e_2)$$

and

$$\lambda_{-1}(h_2)\,\phi_r(\sigma_1(h_2))=\left(\sum_\alpha s_\alpha^2\right)^{(r-2)/2}\left(\sum_\beta s_\beta\right),\qquad r \text{ even.} \tag{6.4}$$

In the symplectic case, we get

$$\lambda_{-1}(e_2)\,\phi_r(\sigma_1(e_2))=\left(\sum_\alpha s_\alpha^2\right)^{(r-2)/2}\sigma_1(-h_1+h_2).$$

But

$$\sigma_1(-h_1+h_2)=\sigma_1((e_1+e_2)(-X))=\sum_\alpha(-1)^{|\beta|}s_\beta,$$

whence

$$\lambda_{-1}(e_2)\,\phi_r(\sigma_1(e_2))=\left(\sum_\alpha s_\alpha^2\right)^{(r-2)/2}\left(\sum_\beta(-1)^{|\beta|}s_\beta\right),\qquad r \text{ even.} \tag{6.5}$$

In particular, for $r=2, 3$, we recover Littlewood's Theorems III–VI of [26], for

6.6. THEOREM.

$$\psi_2(o_\lambda) = D_{\lambda_{-1}(h_2)}[\psi_2(D_{\sigma_1(e_1+e_2)}s_\lambda)] = D_{\lambda_{-1}(h_2)}\psi_2\left(\sum_\alpha s_{\lambda/\alpha}\right).$$

$$\psi_2(sp_\lambda) = D_{\lambda_{-1}(e_2)}\sum_\alpha (-1)^{|\alpha|}\,\psi_2(s_{\lambda/\alpha}).$$

6.7. THEOREM.

$$\psi_3(o_\lambda) = D_{\lambda_{-1}(h_2)}\sum_\alpha \psi_3(D^2_{s_\alpha}s_\lambda),$$

$$\psi_3(sp_\lambda) = D_{\lambda_{-1}(e_2)}\sum_\alpha \psi_3(D^2_{s_\alpha}s_\lambda).$$

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